ERRATUM: "TROPICAL CYCLE CLASSES FOR NON-ARCHIMEDEAN SPACES AND WEIGHT DECOMPOSITION OF DE RHAM COHOMOLOGY SHEAVES"

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In this note, we correct some errors in [Liu20].

1. Remark 2.5

In [Liu20, Remark 2.5], the assignment $\{f_1, \dots, f_p\} \mapsto \frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}f_p}{f_p}$ gives a map $\mathscr{K}^p_X[-p] \to \tau_{\geq p}\Omega^\bullet_X$, not to $\Omega^\bullet_X$. We now give the correct construction of the de Rham cycle class map via sheaves of Milnor K-theory. Let X be a smooth scheme of finite type over a field k of characteristic zero. For every integer $p \geq 0$, denote by $\mathscr{H}^p_X$ the k-linear sheaf on X associated to the presheaf $U \mapsto \mathrm{H}^p_{\mathrm{dR}}(U/k)$, which is nothing but $\Omega^{p,\mathrm{cl}}_X / \mathrm{d}\Omega^{p-1}_X$.

Lemma 1.1. *Let Z be a closed subscheme of X of pure codimension q with $i_Z\colon Z \to X$ the closed immersion.*

1. *If $q > p$, then both $i^!_Z\mathscr{K}^p_X$ and $i^!_Z\mathscr{H}^p_X$ vanish.*
2. *If $q = p$ and Z is regular, then we have canonical isomorphisms $i^!_Z\mathscr{K}^p_X[p] \simeq \mathbb{Q}$ and $i^!_Z\mathscr{H}^p_X[p] \simeq k$ as constant sheaves on Z.*

Proof. By a standard induction argument, we may assume Z regular as well in (1).

For $\mathscr{K}^p_X$, both (1) and (2) follow from the flasque resolution [Liu20, (2.1)].[1] For $\mathscr{H}^p_X$, both (1) and (2) follow from the flasque resolution [BO74, Theorem 4.2].

Here, we have implicitly used the facts that a direct sum of flasque sheaves on a Noetherian topological space is flasque, and that pushforward along a continuous map between Noetherian topological spaces commutes with direct sum. □

Now we take an algebraic cycle c (with rational coefficients) of codimension p with support Z. Let Z' be the regular locus of Z and put $X' \coloneqq X \setminus (Z \setminus Z')$. The cycle c is nothing but an element $c \in \mathrm{H}^0(Z', \mathbb{Q})$. Let $[c]$ be the image of c under the composite map

$$\mathrm{H}^0(Z', \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^p(X', i_{Z'!}i^!_{Z'}\mathscr{K}^p_{X'}) = \mathrm{H}^p_{Z'}(X', \mathscr{K}^p_{X'}) = \mathrm{H}^{2p}_{Z'}(X', \mathscr{K}^p_{X'}[-p]) \to \mathrm{H}^{2p}_{Z'}(X', \tau_{\geq p}\Omega^\bullet_{X'}),$$

where the first isomorphism comes from Lemma 1.1(2). By the purity of the de Rham complex and Lemma 1.1(1), all arrows in the diagram

$$\begin{tikzcd}
\mathrm{H}^{2p}_Z(X, \Omega^\bullet_X) \arrow[r] \arrow[d] & \mathrm{H}^{2p}_{Z'}(X', \Omega^\bullet_{X'}) \arrow[d] \\
\mathrm{H}^{2p}_Z(X, \tau_{\geq p}\Omega^\bullet_X) \arrow[r] & \mathrm{H}^{2p}_{Z'}(X', \tau_{\geq p}\Omega^\bullet_{X'})
\end{tikzcd}$$

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[1] The direct sum in the displayed formula after [Liu20, (2.1)] should be taken over $x \in X^{(q)}$.

are isomorphisms, where horizontal arrows are restrictions. In particular, $[c]$ can be regarded as an element in $\mathrm{H}^{2p}_Z(X,\Omega^\bullet_X)$. The de Rham cycle class $\mathrm{cl}_{\mathrm{dR}}(c)$ of c is simply the image of $[c]$ under the natural map $\mathrm{H}^{2p}_Z(X,\Omega^\bullet_X)\to\mathrm{H}^{2p}(X,\Omega^\bullet_X)=\mathrm{H}^{2p}_{\mathrm{dR}}(X/k)$.

Note that this remark is used in the commutativity of the last diagram in Step 1 of the proof of [Liu20, Theorem 6.6].

2. Proof of Theorem 6.6

In the proof of [Liu20, Theorem 6.6], the first displayed formula in Step 4 is wrong – it was author's misunderstanding of the reference [HK94]. In fact, it is the complex $(\Omega^{\dagger,\bullet}_{X',\mathcal{Y}'},\mathrm{d})$ that admits a Frobenius action (in the derived category of abelian sheaves on $\mathcal{Y}'_s$), which is a consequence of [GK05, Theorem 3.1]. In particular, the Frobenius action induces an action on the associated conjugate spectral sequence. Now, the correct argument is even simpler (than the wrong one). We only need to show that the map

$$\mathrm{H}^{n-p}(\mathcal{Y}'_s,\mathscr{L}^{n-p}_{X',\mathcal{Y}'})\to\mathrm{H}^{2n-2p+1}(\mathcal{Y}'_s,\tau_{\leq n-p-1}\Omega^{\dagger,\bullet}_{X',\mathcal{Y}'}) \tag{2.1}$$

vanishes by the argument at the end of Step 3. The Frobenius action on $(\Omega^{\dagger,\bullet}_{X',\mathcal{Y}'},\mathrm{d})$ induces a weight decomposition of the cohomology sheaf

$$\Omega^{\dagger,q,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{\dagger,q-1}_{X',\mathcal{Y}'}=\bigoplus_{w=0}^{2q}(\Omega^{\dagger,q,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{\dagger,q-1}_{X',\mathcal{Y}'})_w$$

for every $q\geq 0$, under which the image of the canonical map $\mathscr{L}^q_{X',\mathcal{Y}'}\to\Omega^{\dagger,q,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{\dagger,q-1}_{X',\mathcal{Y}'}$ is contained in the subsheaf of (generalized) weight $2q$, by the same argument in Step 5 (but this time applied to the correct object). Then the vanishing of (2.1) follows from the same argument in Step 4 but to the spectral sequence ${}^\dagger\mathrm{E}^{p,q}_r$.

References

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